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Hyperinterpolation on the Sphere at the Minimal Projection Order

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We investigate hyperinterpolation operators based on positive weighted quadrature rules, as introduced by Ian H. Sloan. If the rules are exact of double degree then, independently of the number of their nodes, the operator norms increase at the order of the minimal projections. © 2000 Academic Press

Key Words: hyperinterpolation; minimal projection; sphere covering by caps.

1. INTRODUCTION AND MAIN RESULT

Let $r \in \mathbb{N} \setminus \{1\}$ be a fixed space dimension. For $\mu \in \mathbb{N}_0$ let $\mathbb{P}_\mu$ denote the subspace of $C(S^{r-1})$ consisting of all restrictions onto the unit sphere $S^{r-1} := \{x \in S^{r-1} : |x| = 1\}$ of a r-variate polynomial of (total) degree μ. The elements of $C(S^{r-1})$ are called spherical functions, and those of $\mathbb{P}_\mu$ spherical polynomials of degree μ. We write the euclidean inner product of $x, y \in \mathbb{R}^r$ as xy. Our concern are linear projections

$$L: C(S^{r-1}) \to \mathbb{P}_\mu, \tag{1.1}$$

furnished by the norm

$$\|L\| := \max\{\|Lf\|_\infty : f \in C(S^{r-1}), \|f\|_\infty \leqslant 1\} \tag{1.2}$$

with $\|\cdot\|_\infty$ the uniform norm in $C(S^{r-1})$. The standard measure ω of $C(S^{r-1})$ induces the inner product

$$\langle f, g \rangle = \int_{S^{r-1}} f(x)\, g(x)\, d\omega(x), \qquad f, g \in C(S^{r-1}). \tag{1.3}$$

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The corresponding orthogonal projection

$$\Pi_\mu : C(S^{r-1}) \to \mathbb{P}_\mu \tag{1.4}$$

is described by

$$(\Pi_\mu f)(x) = \int_{S^{r-1}} f(t)\, P_\mu(xt)\, d\omega(t), \qquad f \in C(S^{r-1}), \quad x \in S^{r-1}, \tag{1.5}$$

$P_\mu(xy)$, $x, y \in S^{r-1}$, the reproducing kernel function of $\mathbb{P}_\mu$, i.e.,

$$P_\mu = \alpha_\mu \cdot P_\mu^{((r-1)/2,\,(r-3)/2)}, \tag{1.6}$$

where $P_\mu^{(\alpha,\beta)}$ are the *Jacobi polynomials* of degree μ and indices α, β and where α_μ is defined by the equation

$$P_\mu(1) = \frac{1}{\omega_{r-1}} \dim(\mathbb{P}_\mu).$$

We note that

$$\alpha_\mu = \mathcal{O}(\mu^{(r-1)/2}) \tag{1.7}$$

holds as $\mu \to \infty$. For details see [7].

Daugavet [3] proved, as a generalisation of a result of Berman [2], that Π_μ is minimal for arbitrary $r \geqslant 2$, i.e., that

$$\|\Pi_\mu\| \leqslant \|L\| \tag{1.8}$$

holds for arbitrary projections (1.1) even in the uniform norm.

In case of $r = 2$ the minimal *interpolatory* projection is characterized by an equally spaced distribution of its nodes on the unit circle (trigonometric equioscillation theorem, de Boor and Pinkus [4]). Its uniform norm has been expressed before by Ehlich and Zeller [5] in form of the following finite trigonometric sum,

$$\sum_{j=0}^{2\mu} 1 \Big/ \sin\left(\frac{2j+1}{2\mu+1} \cdot \frac{\pi}{2}\right) \sim \frac{2}{\pi} \ln(\mu), \qquad \mu \to \infty.$$

Summarizing we may state that we possess full knowledge of the minimal interpolatory projection problem in case $r = 2$.

In case of $r \geqslant 3$ we know the order of the minimal projection norm precisely, as the result of Daugavet includes the asymptotics

$$\|\Pi_\mu\| \sim \pi_r \cdot \mu^{(r-2)/2}, \qquad \mu \to \infty, \tag{1.9}$$

where even the constant $\pi_r > 0$ is explicitly known. But we do not know whether this order can be obtained by interpolatory projections. This is the background on which the concept of *hyperinterpolation*, presented by Ian H. Sloan [9], must be valued.

Hyperinterpolation operators $L_\mu\colon C(S^{r-1}) \to \mathbb{P}_\mu$ arise if the orthogonal projection (1.5) is evaluated by a quadrature rule Q_μ,

$$Q_\mu f = \sum_{\nu=1}^{M_\mu} A_\nu f(t_\nu), \tag{1.10}$$

based on nodes $t_1, ..., t_{M_\mu} \in S^{r-1}$ and positive weights $A_1, ..., A_{M_\mu} > 0$, provided it is *exact* of degree 2μ. This requires

$$M_\mu \geqslant N_\mu := \dim(\mathbb{P}_\mu) = \binom{\mu+r-1}{r-1} + \binom{\mu+r-2}{r-1} \sim \frac{2\mu^{r-1}}{(r-1)!} \tag{1.11}$$

as $\mu \to \infty$, cf. [7], where equality holds if and only if Q_μ is a *Gauß-rule*. Such rules do not exist for $(r, \mu) \geqslant (3, 3)$ by a result of Bannai and Damerell [1] on the (non-) existence of tight spherical designs. So hyperinterpolation necessarily requires, in this case, $M_\mu > N_\mu$ nodes, and hence its concept leaves the concept of interpolation. We should mention, however, that $2\mu+1$ equally spaced nodes on the unit circle in $\mathbb{R}^2$ always support a Gauß-rule, such that the corresponding hyperinterpolation and interpolation operators coincide.

In what follows we consider hyperinterpolation operators L_μ, $\mu \in \mathbb{N}_0$, which are based on rules Q_μ which satisfy

$$A_1 > 0, ..., A_{M_\mu} > 0, \tag{1.12}$$

$$Q_\mu f = \int_{S^{r-1}} f(x)\, d\omega(x) \qquad \text{for all} \quad f \in \mathbb{P}_{2\mu}. \tag{1.13}$$

It is well known that sequences of quadrature rules of this kind exist where, in addition, the number of nodes is bounded in form

$$M_\mu \leqslant k_r N_\mu \qquad \text{for all} \quad \mu \in \mathbb{N}_0 \tag{1.14}$$

$k_r > 1$ a constant depending on the space dimension. For a detailed discussion of this question we refer to Sloan [10].

Hyperinterpolation operators have the explicit form

$$L_\mu f(\cdot) = \sum_{\nu=1}^{M_\mu} A_\nu f(t_\nu)\, P_\mu(t_\nu \cdot), \qquad \mu \in \mathbb{N}_0. \tag{1.15}$$

Hence $L_\mu f$ vanishes if f vanishes in all the nodes. Counterexamples show that this need not be true in case of $\Pi_\mu f$. So the operators are different, in general. But hyperinterpolation operators are projections $L_\mu\colon C(S^{r-1}) \to \mathbb{P}_\mu$, and Sloan and Womersley [11] proved that

$$\|L_\mu\| = \mathcal{O}(\mu^{(r-1)/2}), \tag{1.16}$$

holds for $\mu \to \infty$ and arbitrary $r \geqslant 3$, while in the important case of $r = 3$ even the stronger result

$$\|L_\mu\| = \mathcal{O}(\mu^{1/2}), \tag{1.17}$$

$\mu \to \infty$, is valid, supposing that a certain *regularity condition* on the distribution of the nodes is satisfied, which holds, for instance, if *product Gauß rules* are used. This is an important result as the order in (1.17) is the order of the minimal projection norms, cf. (1.9), and hence cannot be improved. As it is possible to satisfy (1.14), these results say that hyperinterpolation is a strong alternative to interpolation–for which we have no comparable results until now. This is challenge for the future.

In this paper we improve the results of Sloan and Womersley by methods we used before in the treatment of other related problems [6, 8]. The main result consists in the observation that the regularity condition is satisfied by its own, and this for arbitrary dimension r. As this was the key for the best order result of Sloan and Womersley for $r = 3$ it is plausible that the following general result holds.

THEOREM 1. *Let $r \in \mathbb{N} \setminus \{1, 2\}$. Then positive constants a_r and b_r exist such that*

$$a_r \mu^{(r-2)/2} \leqslant \|L_\mu\| \leqslant b_r \mu^{(r-2)/2}$$

holds for arbitrary hyperinterpolation operators $L_\mu\colon C(S^{r-1}) \to \mathbb{P}_\mu$, $\mu \in \mathbb{N}$, whose defining quadrature rule Q_μ satisfies (1.12) *and* (1.13).

2. THE REGULARITY CONDITION

In this section we assume $r \in \mathbb{N} \setminus \{1, 2\}$ to be fixed. So, for simplicity, we are allowed not to notify the dependency of quantities on r if the context is admitting this. Recall (1.6), i.e.,

$$P_\mu = \alpha_\mu \cdot P_\mu^{((r-1)/2,\,(r-3)/2)}.$$

Let $\tilde{P}_\mu := P_\mu / P_\mu(1)$ and define Z_ν by

$$Z_\nu(\phi) := \Gamma(\nu+1)\left(\frac{2}{\phi}\right)^\nu J_\nu(\phi),$$

where J_ν is the *Bessel function* of index $\nu := \frac{r-1}{2} \geqslant 1$. The normalisation of these functions is such that $\tilde{P}_\mu(1) = 1 = Z_\nu(0)$, and that

$$\lim_{\mu \to \infty} \tilde{P}_\mu\left(\cos\frac{\phi}{\mu}\right) = Z_\nu(\phi) \tag{2.1}$$

holds uniformly for ϕ in a compact set, see Szegö [12, (8.1.1)].

Next let $x_\mu = \cos\theta_\mu$, $0 < \theta_\mu < \pi$, denote the greatest zero of P_μ, $\mu \in \mathbb{N}$. By the monotonicity of the zeros of the Jacobi polynomials with respect to α and β [12 (6.21.2) and (6.21.3)], $\cos\theta_\mu$ is not less than the greatest zero of the *Legendre polynomial* $P_\mu^{(0,0)}$, and [12, (6.21.5)] implies

$$\theta_\mu \geqslant \frac{1}{2\mu+1}, \qquad \mu \in \mathbb{N}_0. \tag{2.2}$$

So we get $\mu\theta_\mu \geqslant \frac{1}{3}$ and

$$\frac{\theta_\mu}{2} \geqslant \psi_\mu := \frac{\gamma}{\mu}, \qquad \gamma := \frac{\pi}{20}, \tag{2.3}$$

for $\mu \in \mathbb{N}$, independently of the value of r. From (2.1) it follows that

$$\theta_\mu \sim \frac{j_\nu}{\mu}, \qquad \mu \to \infty,$$

holds if j_ν is the lowest positive zero of J_ν or Z_ν, which is the same. By the interlacing property of the zeros of *Bessel functions* we get

$$j_\nu \geqslant \min\{j_{1/2}, j_1\} \geqslant \min\left\{\frac{\pi}{2}, 3.8\right\} = \frac{\pi}{2},$$

see Watson [14, pp. 54 and 748]. It follows that

$$\gamma < j_\nu \tag{2.4}$$

holds, independently of the value of r, again.

We discussed the function Z_ν in [8]. Especially it is monotonically decreasing in the interval $[0, j_\nu]$. Because of (2.1) and (2.4) this yields

$$\tilde{P}_\mu\left(\cos\frac{\gamma}{\mu}\right) \geqslant \frac{1}{2} Z_\nu(\gamma) > \frac{1}{2} Z_\nu(j_\nu) = 0$$

for sufficiently great μ, say $\mu \geqslant \mu_0$. Together with

$$\tilde{P}_\mu\left(\cos\frac{\gamma}{\mu}\right) \geqslant \tilde{P}_\mu\left(\cos\frac{\theta_\mu}{2}\right) > \tilde{P}_\mu(\cos\theta_\mu) = 0,$$

which holds for $\mu = 1, 2, ..., \mu_0$ cf. (2.3), this implies that a constant $c_1 > 0$ exists such that

$$\tilde{P}_\mu\left(\cos\frac{\gamma}{\mu}\right) \geqslant \sqrt{\frac{\omega_{r-1}}{c_1}}, \qquad \mu \in \mathbb{N}, \tag{2.5}$$

is valid. The constant depends on r, though this is not notified.

LEMMA 1. *Let $r \in \mathbb{N}\backslash\{1, 2\}$ be fixed. Assume Q_μ, $\mu \in \mathbb{N}$, satisfies* (1.12) *and* (1.13). *Then, with ψ_μ defined in* (2.3),

$$\sum_{t_j x \geqslant \cos\psi_n} A_j \leqslant c_1 \cdot N_\mu^{-1} \tag{2.6}$$

holds for arbitrary $x \in S^{r-1}$, where c_1 is the constant of (2.5).

Proof. Let $x \in S^{r-1}$ be arbitrary. Then we get

$$\sum_{j=1}^{M_\mu} A_j P_\mu^2(t_j x) = \int_{S^{r-1}} P_\mu^2(tx)\, d\omega(t) = P_\mu(1), \tag{2.7}$$

where we used that $P_\mu(\cdot x)$ reproduces itself at the point $\cdot = x$. Because of (2.3), $P_\mu^2(\cos\phi)$ is monotonically decreasing for $0 \leqslant \phi \leqslant \psi_\mu$, which implies

$$P_\mu^2(\cos\psi_\mu) \cdot \sum_{t_j x \geqslant \cos\psi_\mu} A_j \leqslant P_\mu(1),$$

or, because of (2.3) and (2.5),

$$\sum_{t_j x > \cos\psi_\mu} A_j \leqslant \frac{1}{P_\mu(1)} \cdot \frac{1}{\tilde{P}_\mu^2(\cos(\gamma/\mu))} \leqslant \frac{c_1}{\omega_{r-1}} \cdot \frac{1}{P_\mu(1)}.$$

But

$$\omega_{r-1} P_\mu(1) = N_\mu, \tag{2.8}$$

cf. [7], which finishes the proof.

Remark. In case $r = 3$, but apart of the special value of the constant γ, which is of no importance, inequality (2.6) is the regularity condition of Sloan and Womersley [11]. As it holds by its own, (1.17) is unconditionally valid by their results.

3. COVERING THE SPHERE AND SPHERICAL CAPS

In what follows we investigate coverings of the sphere S^{r-1}, $r \in \mathbb{N} \backslash \{1\}$, by spherical caps

$$K(t, \phi) := \{x \in S^{r-1} \mid tx \geqslant \cos \phi\},$$

$t \in S^{r-1}$, of a fixed radius ϕ, $0 < \phi < \pi$. To this end let $G_\mu(xy)$, $x, y \in S^{r-1}$, be the reproducing kernel function of the space of spherical harmonics of degree $\mu \in \mathbb{N}_0$. It is well known [7] that

$$G_\mu = \text{const} \cdot C_\mu^{(r-2)/2}, \tag{3.1}$$

C_μ^λ the Gegenbauer polynomial of degree μ and index λ, the constant being of no interest, here. For $\mu \in \mathbb{N}$ let us define

$$z_\mu = \cos(\chi_\mu^r), \qquad 0 < \chi_\mu^r < \pi, \tag{3.2}$$

to be the greatest zero of G_μ. We have to notify the dependence of χ_μ^r on r for later use. It is also well known [13, p. 186] that

$$\chi_\mu^r \sim \frac{1}{\mu} \cdot j_{(r-3)/2}, \qquad \mu \to \infty, \tag{3.3}$$

holds, implying that there is a constant $\delta_r > 0$ such that

$$\chi_\mu^r \leqslant \frac{1}{\mu} \delta_r \tag{3.4}$$

is valid for arbitrary $\mu \in \mathbb{N}$.

It was the observation of V. A. Yudin [15], that a *spherical design* always furnishes some covering of the sphere by caps of a certain radius. His idea can be generalized immediately to get the following lemma.

LEMMA 2. *Let* $r \in \mathbb{N} \backslash \{1\}$, $\mu \in \mathbb{N}_0$. *Assume the quadrature rule* (1.10) *satisfies* (1.12) *and* (1.13). *Then*

$$S^{r-1} \subset \bigcup_{j=1}^{M_\mu} K(t_j, \chi_\mu^r). \tag{3.5}$$

We gave a modified proof in [8].

Remark. The essence of Lemma 2 is that the assumptions can be realized under the additional condition of (1.14). In other words, there exists a constant $k_r > 1$ such that the sphere S^{r-1} can be covered by at most $k_r N_\mu$ caps of radius χ_μ^r. Next we shall use Lemma 2 in the construction of coverings for caps $K(t, \phi)$, $t \in S^{r-1}$, $0 < \phi \leqslant \frac{\pi}{2}$, by smaller caps.

LEMMA 3. *Let $r \in \mathbb{N} \setminus \{1, 2\}$ be fixed. Then $q > 0$ exists such that for arbitrary $t \in S^{r-1}$, $\mu \in \mathbb{N}$ and $\phi \in (\psi_\mu, \frac{\pi}{2}]$ the cap $K(t, \phi)$ can be covered by at most $R(\mu, \phi)$ caps of radius ψ_μ, where*

$$R(\mu, \phi) < q \cdot (\mu \sin \phi)^{r-1}$$

holds for $\psi_\mu \leqslant \phi \leqslant \frac{\pi}{2}$. $R(\mu, \phi)$ does not depend on the choice of t.

Proof. We generalize a construction, which Sloan and Womersley [11] used in the proof of (1.17) in case of $r = 3$, to arbitrary dimensions, where Lemma 2 will be the important tool.

So let $r \in \mathbb{N} \setminus \{1, 2\}$ be arbitrary, but fixed, and let $\mu \in \mathbb{N}$. By Lemma 2 we know how S^{r-2} can be covered by caps of radius χ_μ^{r-1}, or greater than that.

Now let $t \in S^{r-1}$ be arbitrary. We define a partition of the interval $[0, \frac{\pi}{2}]$ by introducing the points

$$\phi_j := \psi_\mu \cdot j = \frac{\pi}{20\mu} \cdot j, \qquad j = 1, ..., 10\mu, \tag{3.6}$$

see (2.3). Obviously we get, because of $\phi_1 = \psi_\mu$,

$$K(t, \phi) \subset K(t, \psi_\mu) \qquad \text{for} \quad 0 < \phi \leqslant \phi_1, \tag{3.7}$$

i.e., one single cap of radius ψ_μ is covering $K(t, \phi)$ in this case.

We are now going to study coverings of caps $K(t, \phi)$ where $\phi_1 < \phi \leqslant \frac{\pi}{2}$. In the beginning we follow still the idea of Sloan and Womersley and introduce the spherical collars

$$B_j := \{x \in S^{r-1} \mid \cos \phi_{j+1} \leqslant tx \leqslant \cos \phi_j\}$$

for $j = 1, ..., 10\mu - 1$. But after this we define

$$S_t^{r-2} := \{u \in S^{r-1} \mid ut = 0\},$$

which is a unit sphere as S^{r-1} is, but of one dimension less, and assume that $\kappa \in \mathbb{N}$ is an arbitrary number, but so great that

$$\frac{2\pi}{\kappa} \delta_{r-1} \leqslant \sqrt{2} - 1. \tag{3.8}$$

Because of Lemma 2 we can cover S_t^{r-2} for every fixed $j \in \{1, ..., 10\mu - 1\}$ by $M_{\kappa j}^{r-1}$ caps with radius $\chi_{\kappa j}^{r-1}$, say with centers $u_{j,k} \in S_t^{r-2}$, this means by

$$K(u_{j,k}, \chi_{\kappa j}^{r-1}), \qquad k = 1, ..., M_{\kappa j}^{r-1}, \tag{3.9}$$

where according to the Remark, but with $r-1$ instead of r, κj instead of μ and with a suggestive change in the notation,

$$M_{\kappa j}^{r-1} \leqslant k_{r-1} N_{\kappa j}^{r-1} \tag{3.10}$$

holds for some constant $k_{r-1} > 1$ where

$$N_\tau^{r-1} = \binom{\tau + r - 2}{r-2} + \binom{\tau + r - 3}{r-2} \tag{3.11}$$

for $\tau \in \mathbb{N}_0$, see (1.11). In addition we define $\bar{\phi}_j := \frac{1}{2}(\phi_j + \phi_{j+1})$ and introduce the nodes

$$t_{j,k} := \cos \bar{\phi}_j \cdot t + \sin \bar{\phi}_j \cdot u_{j,k} \in B_j \tag{3.12}$$

for $k = 1, ..., M_{\kappa j}^{r-1}$.

After these constructions let $x \in B_j$, say

$$x = \cos \xi \cdot t + \sin \xi \cdot u, \qquad u \in S_t^{r-2}, \quad \phi_j \leqslant \xi \leqslant \phi_{j+1}, \tag{3.13}$$

where u is uniquely determined by x. Obviously, by the covering property of the caps (3.9) there is a point $u_{j,k}$ such that $u \in K(u_{j,k}, \chi_{\kappa j}^{r-1})$, i.e., that

$$uu_{j,k} \geqslant \cos \chi_{\kappa j}^{r-1} \geqslant \cos\left(\frac{1}{\kappa j} \delta_{r-1}\right), \tag{3.14}$$

the last inequality following from (3.4) and (3.8), where a consequence of (3.8) is that $\frac{1}{\kappa j}\delta_{r-1} < \frac{\pi}{2}$ such that the cosine is monotonically decreasing in the interval of interest.

We want to estimate the distance between x and $t_{j,k}$. To this end we introduce the point

$$\bar{t}_j := \cos \bar{\phi}_j \cdot t + \sin \bar{\phi}_j \cdot u \in B_j, \tag{3.15}$$

and, by the triangle inequality,

$$|x - t_{j,k}| \leqslant |x - \bar{t}_j| + |\bar{t}_j - t_{j,k}|. \tag{3.16}$$

First we get from (3.13) and (3.15), together with (3.6),

$$|x - \bar{t}_j|^2 = (\cos \xi - \cos \bar{\phi}_j)^2 + (\sin \xi - \sin \bar{\phi}_j)^2 = 4 \cdot \sin^2 \frac{\xi - \bar{\phi}_j}{2}$$

and hence

$$|x-\bar{t}_j| \leqslant 2 \sin \frac{\psi_\mu}{4}. \tag{3.17}$$

Next we get from (3.15) and (3.12)

$$\begin{aligned} |\bar{t}_j - t_{j,k}|^2 &= \sin^2 \bar{\phi}_j \cdot |u-u_{j,k}|^2 = 2 \sin^2 \bar{\phi}_j \cdot (1-uu_{j,k}) \\ &\leqslant 2 \sin^2 \bar{\phi}_j \cdot \left(1-\cos\left(\frac{1}{\kappa j}\,\delta_{r-1}\right)\right) \\ &= 4 \sin^2 \bar{\phi}_j \cdot \sin^2\left(\frac{1}{2\kappa j}\,\delta_{r-1}\right), \end{aligned}$$

the inequality following from (3.14). So we obtain

$$\begin{aligned} |\bar{t}_j - t_{j,k}| &\leqslant \bar{\phi}_j \cdot \frac{1}{\kappa j} \cdot \delta_{r-1} = \left(1+\frac{1}{2j}\right) \cdot \psi_\mu \cdot \frac{1}{\kappa}\,\delta_{r-1} \\ &\leqslant 2\left(\frac{2}{\pi} \cdot \frac{\psi_\mu}{4}\right) \cdot \left(\frac{2\pi}{\kappa}\,\delta_{r-1}\right) \\ &\leqslant 2\left(\sin \frac{\psi_\mu}{4}\right) \cdot \left(\frac{2\pi}{\kappa}\,\delta_{r-1}\right). \end{aligned}$$

By the choice of κ, cf. (3.8), we get the estimate

$$|\bar{t}_j - t_{j,k}| \leqslant 2(\sqrt{2}-1) \sin \frac{\psi_\mu}{4}, \tag{3.18}$$

independently of the value of j. Together with (3.16), (3.17) this yields

$$|x - t_{j,k}| \leqslant 2\sqrt{2} \sin \frac{\psi_\mu}{4},$$

$$1 - xt_{j,k} = \frac{1}{2}\,|x-t_{j,k}|^2 \leqslant 4 \sin^2 \frac{\psi_\mu}{4} = 2\left(1-\cos \frac{\psi_\mu}{2}\right)$$

and finally

$$xt_{j,k} \geqslant -1 + 2 \cos \frac{\psi_\mu}{2} \geqslant -1 + 2\cos^2 \frac{\psi_\mu}{2} = \cos \psi_\mu,$$

which is the same as

$$x \in K(t_{j,k}, \psi_\mu). \tag{3.19}$$

This means that B_j is covered by the caps

$$K(t_{j,k}, \psi_\mu), \qquad k = 1, ..., M_{\kappa j}^{r-1}.$$

Now we return to our original problem, namely to cover $K(t, \phi)$ under the condition $(\psi_\mu =)\ \phi_1 < \phi \leqslant \frac{\pi}{2}$. But in this case we get

$$K(t, \phi) \subset K(t, \psi_\mu) \cup B_1 \cup \cdots \cup B_i,$$

provided we choose

$$i := \left\lceil \frac{\phi}{\psi_\mu} \right\rceil - 1. \tag{3.20}$$

It follows from the result of above that $K(t, \phi)$ can be covered by $R(\mu, \phi)$ caps of radius ψ_μ where

$$R(\mu, \phi) \leqslant 1 + \sum_{j=1}^{i} M_{\kappa j}^{r-1}.$$

In view of (3.10), (3.11) this implies

$$R(\mu, \phi) \leqslant 1 + k_{r-1} \sum_{j=1}^{i} \left\{ \binom{\kappa j + r - 2}{r - 2} + \binom{\kappa j + r - 3}{r - 2} \right\}.$$

The argument of the sum is a polynomial of degree $r-2$ with respect to j. So there is a constant q' such that

$$R(\mu, \phi) \leqslant q' \cdot i^{r-1}$$

holds independently of the value of i, and thus of ϕ. Together with (3.20) and (2.3) this yields

$$R(\mu, \phi) \leqslant q' \cdot \left(\frac{\phi}{\psi_\mu} \right)^{r-1} = q' \cdot \gamma^{r-1} \cdot (\mu\phi)^{r-1}.$$

So we get, finally,

$$R(\mu, \phi) \leqslant q \cdot (\mu \sin \phi)^{r-1}$$

with the constant $q := q' \gamma^{1-r} (\frac{\pi}{2})^{r-1}$. This finishes the proof.

4. PROOF OF THEOREM 1

Again let $r \in \mathbb{N}\backslash\{1, 2\}$ be fixed and let L_μ, $\mu \in \mathbb{N}$, be as in Theorem 1. L_μ is a projection, so

$$\|L_\mu\| \geqslant \|\Pi_\mu\| \sim \pi_r \cdot \mu^{(r-2)/2}$$

holds for $\mu \to \infty$, (1.9), and the existence of a proper constant a_r follows.

Next we get from (1.15) the upper estimate

$$\|L_\mu\| \leqslant \max\left\{\sum_{\nu=1}^{M_\mu} A_\nu \, |P_\mu(t_\nu x)| \colon x \in S^{r-1}\right\}.$$

It follows that

$$\|L_\mu\| \leqslant S_\mu^+ + S_\mu^- \tag{4.1}$$

holds if we define

$$S_\mu^+ := \max\left\{\sum_{t_\nu x \geqslant 0} A_\nu \, |P_\mu(t_\nu x)| \colon x \in S^{r-1}\right\},$$

$$S_\mu^- := \max\left\{\sum_{-t_\nu x \geqslant 0} A_\nu \, |P_\mu(t_\nu x)| \colon x \in S^{r-1}\right\}.$$

S_μ^- has the same form as S_μ^+, except that it belongs to the hyperinterpolation operator $\bar{L}_\mu$ which arises if Q_μ is applied to the representation of $(\Pi_\mu f)(x)$ in (1.5) where the integrator t is replaced by $-t$. This means that in (1.15) the nodes t_ν are replaced by $\bar{t}_\nu = -t_\nu$. The arising quadrature rule $\bar{Q}_\mu$ satisfies (1.12) and (1.13), again. So it suffices to estimate S_μ^+, only.

To this end we estimate the expression

$$\sum_{t_\nu t \geqslant 0} A_\nu \, |P_\mu(t_\nu t)|$$

for a fixed $t \in S^{r-1}$ to above, where it is helpful to use the *weighted nodes counting function*

$$A(\phi) := \sum_{t_\nu t \geqslant \cos\phi} A_\nu, \qquad 0 \leqslant \phi \leqslant \pi,$$

which is nonnegative and monotonically not decreasing. Note that

$$A(\pi) = Q_\mu 1 = \omega_{r-1} \tag{4.2}$$

holds as Q_μ is exact for the constants. In addition, according to Lemma 3 we can cover $K(t, \phi)$, $\psi_\mu \leqslant \phi \leqslant \frac{\pi}{2}$, by $R(\mu, \phi) < q(\mu \sin \phi)^{r-1}$ caps of radius ψ_μ. By Lemma 1 the contribution of each cap of this size to $A(\phi)$ is at most $c_1 N_\mu^{-1}$. Together this yields

$$A(\phi) \leqslant q c_1 N_\mu^{-1} (\mu \sin \phi)^{r-1}, \qquad \psi_\mu \leqslant \phi \leqslant \frac{\pi}{2}.$$

So, in view of (1.11), a constant c_2 exists such that

$$A(\phi) \leqslant c_2 (\sin \phi)^{r-1}, \qquad \psi_\mu \leqslant \phi \leqslant \frac{\pi}{2}, \tag{4.3}$$

holds for arbitrary $\mu \in \mathbb{N}$. Now we get, with the definitions of Section 3,

$$\sum_{t_\nu t \geqslant 0} A_\nu \, |P_\mu(t_\nu t)| \leqslant P_\mu(1) \sum_{t_\nu \in K(t, \psi_\mu)} A_\nu + \int_{\psi_\mu}^{\pi/2} |P_\mu(\cos \phi)| \, dA(\phi), \tag{4.4}$$

where the integral has to be defined in the sense of Riemann and Stieltjes. We used a similar representation earlier [6] in the estimation of the row sums of some fundamental matrices.

The first term can be estimated by the help of Lemma 1 together with (2.8):

$$P_\mu(1) \sum_{t_\nu \in K(t, \psi_\mu)} A_\nu \leqslant \frac{c_1}{\omega_{r-1}}. \tag{4.5}$$

For the estimate of the second term we first recall that $\psi_\mu = \frac{\gamma}{\mu}$ where $\gamma = \frac{\pi}{20}$, cf. (2.3). Using Szegö [12, (7.32.5)], (1.6), (1.7) and the well known equation

$$P_\mu^{((r-1)/2, (r-3)/2)}(1) = \binom{\mu + \frac{r-1}{2}}{\mu},$$

we find that there is a positive constant c_3 such that

$$(\sin \phi)^{r/2} \, |P_\mu(\cos \phi)| \leqslant c_3 \mu^{(r-2)/2}, \qquad \psi_\mu \leqslant \phi \leqslant \frac{\pi}{2}, \tag{4.6}$$

holds for arbitrary $\mu \in \mathbb{N}$. So we get, using in what follows (4.6), integration by parts, (4.3) (to obtain the third inequality), and (4.2),

$$\begin{aligned}
&\int_{\psi_\mu}^{\pi/2} |P_\mu(\cos\phi)|\, dA(\phi) \\
&\quad \leqslant c_3 \mu^{(r-2)/2} \cdot \int_{\psi_\mu}^{\pi/2} (\sin\phi)^{-r/2}\, dA(\phi) \\
&\quad \leqslant c_3 \mu^{(r-2)/2} \left\{ A\left(\frac{\pi}{2}\right) + \frac{r}{2} \int_{\psi_\mu}^{\pi/2} A(\phi)\,(\sin\phi)^{-(r+2)/2} \cos\phi\, d\phi \right\} \\
&\quad \leqslant c_3 \mu^{(r-2)/2} \left\{ A(\pi) + \frac{r}{2} c_2 \int_0^{\pi/2} (\sin\phi)^{r-1} (\sin\phi)^{-(r+2)/2} \cos\phi\, d\phi \right\} \\
&\quad = c_3 \mu^{(r-2)/2} \left\{ \omega_{r-1} + \frac{r}{2} c_2 \int_0^{\pi/2} (\sin\phi)^{(r-4)/2} \cos\phi\, d\phi \right\} \\
&\quad = c_3 \mu^{(r-2)/2} \left\{ \omega_{r-1} + \frac{r}{r-2} c_2 \right\}.
\end{aligned}$$

Now let

$$b_r := 2\left(\frac{c_1}{\omega_{r-1}} + \frac{r}{r-2} c_2 c_3 + \omega_{r-1} c_3\right).$$

Then, inserting in (4.4) our result together with (4.5), we get

$$\sum_{t_\nu t \geqslant 0} A_\nu\, |P_\mu(t_\nu t)| \leqslant \tfrac{1}{2} b_r \mu^{(r-2)/2}, \qquad \mu \in \mathbb{N}.$$

As t was arbitrary, this yields

$$S_\mu^+ \leqslant \tfrac{1}{2} b_r \mu^{(r-2)/2}, \qquad \mu \in \mathbb{N}.$$

For S_μ^- the same bound holds. So we get from (4.1)

$$\|L_\mu\| \leqslant b_r \mu^{(r-2)/2}, \qquad \mu \in \mathbb{N},$$

and Theorem 1 is proved.

Remark. We should point out that the existence of quadrature rules for S_t^{r-2} which satisfy (1.14) with $r-1$ instead of r was helpful in the proof of Lemma 3. Actually the constant q depend on k_{r-1}, and so does b_r in Theorem 1.

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